Consistency, Validity, and Infinite Models

The following three claims all hold true of the natural numbers 0, 1, 2, ….

> **1. No number is less than itself.**
>
> **2. For any numbers x, y, and z: if x is less than y, and y is less than z, then x is less than z.**
>
> **3. Each number has some number or other that it's less than.**

We can translate these claims into the formal language of Chapter Six, using this translation key

$\mathbf{G^2ab}$: a is less than b

1. $\forall x \sim G^2xx$
(No number is less than itself.)

2. $\forall x\ \forall y\ \forall x\ ((G^2xy \wedge G^2yz) \rightarrow G^2xz)$
(For any numbers x, y, and z: if x is less than y, and y is less than z, then x is less than z.)

3. $\forall x\ \exists y\ G^2xy$
(Each number has some number or other that it's less than.)

We then ask: do these three sentences together form a **consistent** set of claims – or do they, on the contrary, contradict one another, forming an inconsistent set of claims?

Equivalently: instead of asking about the consistency of three sentences, we could wedge these three sentences into a single triple-barreled conjunction, and ask of it: **is this a consistent sentence**?

$$((\underline{\forall x \sim G^2xx} \wedge \underline{\forall x\ \forall y\ \forall x\ ((G^2xy \wedge G^2yz) \rightarrow G^2xz)}) \wedge \underline{\forall x\ \exists y\ G^2xy})$$

That's a silly question: since the natural numbers make all three of these sentences true (interpreting "G^2" as "less than"), there's clearly a model making the whole conjunction true, hence establishing the consistency of this conjunction.

But the domain of that model is the entire set of natural numbers: {0, 1, 2, …}. Since there are an infinite number of them, the set of natural numbers forms an **infinite domain** for such a model.

Now there's nothing in the definition of "model" or "domain" that rules out having an infinite domain in a model. But as a matter of fact, in Chapter Five we never needed to appeal to infinite domains to settle questions about consistency or validity. There, if a sentence was consistent there was a finite model (a model with a finite domain of objects) that showed that; and if an argument was invalid there was a finite model serving as validity counterexample.

Can we build a finite model showing that our three-way conjunction is consistent – a model making all three parts of the conjunction true?

Suppose we start with just one object in the domain, the number Zero.[1]

$\mathbb{D}$: {**0**}

a: **0**
$\mathbf{G^2}$: { }

Since the extension of "$\mathbf{G^2}$" is empty, no number here is less than any number. But to make the third part of the conjunction true, Zero needs to be less than some number.

3. ∀x ∃y $\mathbf{G^2}$xy
(Each number has some number or other that it's less than.)

[1] We've previously reserved "0" to mean false in the semantics, and "1" to mean true. But to make this example easier to read, we here use "0" and "1" in their normal numerical use.

And we can't have Zero be less than itself; for that would make the first part of the conjunction false: no number is less than itself.

1. $\mathbf{\forall x \sim G^2xx}$
(No number is less than itself.)

So we need some other number for Zero to be less than – say, the number One. And then we specify, in the extension of "$\mathbf{G^2}$," that Zero is less than One.

$\mathbb{D}$: {**0**, **1**}

a: **0** **b**: **1**
$\mathbf{G^2}$: {<**0**, **1**>}

But to make the third part of the conjunction true in the model, One needs to be less than some number in the model.

3. $\mathbf{\forall x \exists y\, G^2xy}$
(Each number has some number or other that it's less than.)

And to make the first part of the conjunction true, One can't be less than itself.

1. $\mathbf{\forall x \sim G^2xx}$
(No number is less than itself.)

Could Zero serve as the number which One is less than? No: for in that case One would be less than Zero, and we've already said that Zero is less than One.

$\mathbf{(G^2ba \wedge G^2ab)}$
(One is less than Zero and Zero is less than One.)

But the second part of the conjunction has, as an instance, the claim that if Zero is less than One and One is less than Zero, then Zero is less than Zero.

2. $\forall x\ \forall y\ \forall x\ ((G^2xy \wedge G^2yz) \rightarrow G^2xz)$
(For any numbers x, y, z: if x is less than y, and y is less than z, then x is less than z.)

Instance: **$((G^2ab \wedge G^2ba) \rightarrow G^2aa)$**
(If Zero is less than One and One is less than Zero, then Zero is less than Zero.)

But having Zero less than itself makes the first part of the conjunction false.

1. **$\forall x \sim G^2xx$**
(No number is less than itself.)

So since the third part of the conjunction requires One to be less than some number in the domain, and that number can't be Zero or One, we need to add another number to the domain – say, Two. (And we note that One is less than Two.)

$\mathbb{D}$: {**0**, **1**, **2**}

a: **0** **b**: **1** **c**: **2**
G^2: {<**0**, **1**>**,** <**1**, **2**>}

Moreover the second part of the conjunction has as an instance that if Zero is less than One and One is less than Two, then Zero is less than two.

2. $\forall x\ \forall y\ \forall x\ ((G^2xy \wedge G^2yz) \rightarrow G^2xz)$
(For any numbers x, y, z: if x is less than y, and y is less than z, then x is less than z.)

Instance: **$((G^2ab \wedge G^2ba) \rightarrow G^2aa)$**
(If Zero is less than One and One is less than Zero, then Zero is less than Zero.)

So we need to add <**0**, **2**> to the extension of "**G^2**" to avoid inconsistency.[2]

$\mathbb{D}$: {**0**, **1**, **2**}

a: **0** **b**: **1** **c**: **2**
G^2: {<**0**, **1**>, <**1**, **2**>, <**0**, **2**>}

Since the third part of the conjunction requires every number to be less than some number (or other), we need to have Two be less than some number in the domain.

3. $\forall$x $\exists$y G^2xy
(Each number has some number or other that it's less than.)

And to make the first part of the conjunction true, that number can't be Two itself.

1. **$\forall$x ~G^2xx**
(No number is less than itself.)

Could some other number already in the domain serve as the number which Two is less than? No: if, for instance, Zero is the number that Two is less than, then – since Zero is already less than Two – the following conjunction would be true.

(G^2ca $\wedge$ G^2ac)
(Two is less than Zero and Zero is less than Two.)

In that case, by the second part of the conjunction, Two would be less than itself.

2. $\forall$x $\forall$y $\forall$x ((G^2xy $\wedge$ G^2yz) $\rightarrow$ G^2xz)
(For any numbers x, y, z: if x is less than y, and y is less than z, then x is less than z.)

Instance: **((G^2ca $\wedge$ G^2ac) $\rightarrow$ G^2cc)**
(If Two is less than Zero and Zero is less than Two, then Two is less than Two.)

[2] If <**0**, **1**> and <**1**, **2**> are in the extension of "**G^2**" but <**0**, **2**> isn't, then the sentence "**((G^2ab $\wedge$ G^2bc)**" will be true in this model but "**G^2ac**" will be false – making the conditional "**((G^2ab $\wedge$ G^2bc) $\rightarrow$ G^2ac)**" false. Since this conditional is an instance of "**2. $\forall$x $\forall$y $\forall$x ((G^2xy $\wedge$ G^2yz) $\rightarrow$ G^2xz)**", that universal sentence will be false in this model – making the whole three-way conjunction false in this model.

Two being less than itself would make the first part of the conjunction false.

1. $\forall x \sim G^2xx$
(No number is less than itself.)

Likewise if Two were less than One, then – since One is already less than Two – the following conjunction would be true in this model.

$(G^2cb \wedge G^2bc)$
(Two is less than One and One is less than Two.)

In that case, by the second part of the conjunction, Two would again be less than itself.

2. $\forall x\ \forall y\ \forall x\ ((G^2xy \wedge G^2yz) \rightarrow G^2xz)$
(For any numbers x, y, z: if x is less than y, and y is less than z, then x is less than z.)

Instance: $((G^2cb \wedge G^2bc) \rightarrow G^2cc)$
(If Two is less than One and One is less than Two, then Two is less than Two.)

And that too would make the first part of the conjunction false.

1. $\forall x \sim G^2xx$
(No number is less than itself.)

So to satisfy the requirement that Two be less than some number, we need to add another number to the domain – say, Three. (And we add that Two is less than Three.)

$\mathbb{D}$: {**0**, **1**, **2**, **3**}

a: **0** b: **1** c: **2** d: **3**
G^2: {<**0**, **1**>, <**1**, **2**>, <**2**, **3**>}

Again (on pain of inconsistency) we must also add to the extension of "$\mathbf{G^2}$" that Zero and One are each less than Three.

$\mathbb{D}$: {**0**, **1**, **2**, **3**}

a: **0** **b**: **1** **c**: **2** **d**: **3**
$\mathbf{G^2}$: {<**0**, **1**>, <**1**, **2**>, <**0**, **3**>, <**1**, **3**>, <**2**, **3**>}

But then by the third part of the conjunction, Three must be less than some number in the model.

3. $\mathbf{\forall x\ \exists y\ G^2xy}$
(Each number has some number or other that it's less than.)

At this point we can see that this reasoning will just keep looping around, add more and more numbers in an attempt to satisfy all three parts of our conjunction. With each added number, the third part of the conjunction requires that there be some number which that added number be less than. But the first and second parts of the conjunction conspire to ensure that neither the added number, nor any number less than it, serve this role. So a further number is added to the model, and we repeat.

The upshot: we know there's a model making this conjunction true – namely, the natural numbers under the 'less than' relation.

$$\mathbf{((\underline{\forall x \sim G^2xx} \wedge \underline{\forall x\ \forall y\ \forall x\ ((G^2xy \wedge G^2yz) \rightarrow G^2xz)}) \wedge \underline{\forall x\ \exists y\ G^2xy})}$$

But the set of natural numbers forms an **infinite domain**. And we see now that any attempt to make this conjunction true with a **finite domain** is bound to fail. So we've found a Chapter Six sentence which can only be shown consistent using an infinite model. We could say that this conjunction is **consistent, but not finitely consistent**.

Should consistency seem uninteresting, let's tie this point to the matter of argument validity. For an argument is invalid just in case there's a possible situation – a model – making all the premises true and the conclusion false. But then if this triple-barreled conjunction serves as the premise of an invalid argument, the validity counterexample for that argument will be **a model making this conjunction true** and the conclusion false. Since only an infinite model will make this conjunction true, **only an infinite model will qualify as a validity counterexample** for that argument.

The following crazy argument, for instance, has only infinite models as validity counterexample.

1. $((\underline{\forall x \sim G^2xx} \wedge \underline{\forall x\ \forall y\ \forall x\ ((G^2xy \wedge G^2yz) \rightarrow G^2xz)}) \wedge \underline{\forall x\ \exists y\ G^2xy})$

$\therefore$ **X**

So the language of Chapter Six is fundamentally different from the formal languages of previous chapters. **For the first time, our formal language allows us to state arguments which have only infinite models as counterexample**.

That spells drastic consequences for truth trees. For to construct the infinite counterexample for this argument, the truth tree must go through an infinite number of steps. But a test of validity that needs to go through an infinite number of steps is a test that never comes to an end. That means: **for the above argument the truth tree test of validity will go on without end** – never, at any finite number of steps, either closing or providing a counterexample.

If a test of validity always tells us, in a finite number of steps, that a valid argument is valid; and tells us, in a finite number of steps that an invalid argument is invalid then that test of validity is said to be a **decision procedure**, and matters of validity are in that case **decidable**.

What we see here is that, for some invalid Chapter Six arguments, the truth tree test of validity will never, after any finite number of steps, either close or yield a validity counterexample. That is: the truth tree test of validity fails to be a decision procedure for the Chapter Six language. Hence in the language of Chapter Six the question of validity is **undecidable**: for some invalid cases the test will never end, and so never (in any finite number of steps) yield a verdict at all.